Dominated Strategies

A strategy is said to be dominated by a second strategy if the second strategy always results in at least as good an outcome for the player, no matter what strategy the other player chooses, and results in a better outcome for at least one of the opponent's strategies. We call the inferior strategy a dominated strategy. A strategy which dominates all others is called a dominant strategy. It must be unique.

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- Recall our example from the previous section, where two fitness companies were trying to decide where to set up their new Gymnasiums. The payoff matrix is shown below
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		Fitness	Indiana
		First Neighborhood	Second Neighborhood
Get Up 'n Go	**First Neighborhood**	(1500, 3500)	(5000, 3000)
	Second Neighborhood	(3000, 5000)	(900, 2100)

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- Recall our example from the previous section, where two fitness companies were trying to decide where to set up their new Gymnasiums. The payoff matrix is shown below
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		Fitness First Neighborhood	Indiana Second Neighborhood
Get Up 'n Go	First Neighborhood	(1500, 3500)	(5000, 3000)
	Second Neighborhood	(3000, 5000)	(900, 2100)

- Does either business have any dominated or dominant strategies?

Dominated Strategies

The strategy “Second Nbhd.” is dominated by the strategy “First Nbhd.” for Fitness Indiana because the strategy “First Nbhd.” always results in at least as good an outcome for Fitness Indiana, no matter what strategy Get Up and Go chooses, and results in a better outcome for at least one of Get Up and Go’s strategies(in fact all in this case). The strategy “Second Nbhd.” is a dominated strategy for Fitness Indiana. Because the strategy “First Nbd.” dominates all others, it is called a dominant strategy for Fitness Indiana.

		Fitness	**Indiana**
		First Neighborhood	**Second Neighborhood**
Get Up ’n Go	**First Neighborhood**	(1500,3500)	(5000, 3000)
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Dominated Strategies

The strategy "Second Nbhd." is dominated by the strategy "First Nbhd." for Fitness Indiana because the strategy "First Nbhd." always results in at least as good an outcome for Fitness Indiana, no matter what strategy Get Up and Go chooses, and results in a better outcome for at least one of Get Up and Go's strategies(in fact all in this case). The strategy "Second Nbhd." is a dominated strategy for Fitness Indiana. Because the strategy "First Nbd." dominates all others, it is called a dominant strategy for Fitness Indiana.

		Fitness	Indiana
		First Neighborhood	**Second Neighborhood**
Get Up 'n Go	**First Neighborhood**	(1500,3500)	(5000, 3000)
	Second Neighborhood	(3000, 5000)	(900, 2100)

- It would be unwise of Fitness Indiana to choose the strategy "Second Nbhd." here. Since the pay-off matrix is public information, both business' know that Fitness Indiana will not go for this option and it can be eliminated from the matrix.

Reduced Payoff Matrix

After we remove the dominated strategy for Fitness Indiana, we get a new pay-off matrix:

		Fitness Indiana
		First Neighborhood
Get Up 'n Go	**First Neighborhood**	(1500, 3500)
	Second Neighborhood	(3000, 5000)

Reduced Payoff Matrix

After we remove the dominated strategy for Fitness Indiana, we get a new pay-off matrix:

		Fitness Indiana
		First Neighborhood
Get Up 'n Go	**First Neighborhood**	(1500, 3500)
	Second Neighborhood	(3000, 5000)

- In this new matrix, are there any dominated or dominant strategies for Get Up and Go?

Reduced Payoff Matrix

After we remove the dominated strategy for Fitness Indiana, we get a new pay-off matrix:

		Fitness Indiana
		First Neighborhood
Get Up 'n Go	**First Neighborhood**	(1500, 3500)
	Second Neighborhood	(3000, 5000)

Reduced Payoff Matrix

After we remove the dominated strategy for Fitness Indiana, we get a new pay-off matrix:

		Fitness Indiana
		First Neighborhood
Get Up 'n Go	**First Neighborhood**	(1500, 3500)
	Second Neighborhood	(3000, 5000)

- We see that for Get Up and Go, the strategy "Second Neighborhood" dominates the strategy "First Neighborhood" and because we have only two strategies, the strategy "Second Neighborhood" is a dominant strategy.

Reduced Payoff Matrix

After we remove the dominated strategy for Fitness Indiana, we get a new pay-off matrix:

		Fitness Indiana
		First Neighborhood
Get Up 'n Go	First Neighborhood	(1500, 3500)
	Second Neighborhood	(3000, 5000)

- We see that for Get Up and Go, the strategy "Second Neighborhood" dominates the strategy "First Neighborhood" and because we have only two strategies, the strategy "Second Neighborhood" is a dominant strategy.
- Since both business' want to maximize the number of customers the get, it would be foolish for Get Up and Go to set up their gym in the "First Neighborhood" and we might as well further reduce the payoff matrix by removing this strategy.

Reduced Payoff Matrix

Finally we are left with the reduced payoff matrix

		Fitness Indiana
		First Neighborhood
Get Up 'n Go	**Second Neighborhood**	(3000, 5000)

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Reduced Payoff Matrix

Finally we are left with the reduced payoff matrix

		Fitness Indiana
		First Neighborhood
Get Up 'n Go	Second Neighborhood	(3000, 5000)

- We see that if both players seek to maximize the number of customers they get and both have complete information about the payoff matrix, Get Up and Go will choose the Second Neighborhood and Fitness Indiana will choose the First Neighborhood.

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Reduced Payoff Matrix

Finally we are left with the reduced payoff matrix

		Fitness Indiana
		First Neighborhood
Get Up 'n Go	**Second Neighborhood**	(3000, 5000)

- We see that if both players seek to maximize the number of customers they get and both have complete information about the payoff matrix, Get Up and Go will choose the Second Neighborhood and Fitness Indiana will choose the First Neighborhood.
- Since there are no dominated strategies in this final matrix with only one strategy for each player, it is called a reduced Payoff matrix.

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Reduced Payoff Matrix

The Reduced Payoff Matrix of a game is a submatrix of the game where dominated strategies have been eliminated in one or more stages. It gives the relevant portion of the original payoff matrix under the assumption of best play by both players.

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- The reduced payoff matrix is not always a 1×1 matrix, it may be larger. Its distinguishing characteristic is that neither player has a dominated strategy in the reduced matrix.

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The Reduced Payoff Matrix of a game is a submatrix of the game where dominated strategies have been eliminated in one or more stages. It gives the relevant portion of the original payoff matrix under the assumption of best play by both players.

- The reduced payoff matrix is not always a 1×1 matrix, it may be larger. Its distinguishing characteristic is that neither player has a dominated strategy in the reduced matrix.
- If the reduced payoff matrix has a single strategy for each player, neither player has an incentive to change strategy if the other player sticks with the strategy given in the reduced matrix. Thus this is a point of equilibrium.

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Equilibrium Point

An Equilibrium Point of a game is a pair of strategies such that neither player has any incentive to change strategies if the other player stays with their current strategy.

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- In the example from Dutta, we found that the payoff matrices for the swimmer Rogers were different depending on whether the IOC performed a drug test or not. The pay off matrices are given below where b is a very large number:

No Testing (Probabilities)

		Carter	
		d	**n**
Rogers	**d**	0.5	1
	n	0	0.5

IOC Testing (Expected Payoff)

		Carter	
		d	**n**
Rogers	**d**	$(-\frac{b}{2}, -\frac{b}{2})$	$(-\frac{b}{2}, 0)$
	n	$(0, -\frac{b}{2})$	$(0, 0)$

.

Equilibrium Point

An Equilibrium Point of a game is a pair of strategies such that neither player has any incentive to change strategies if the other player stays with their current strategy.
Rewriting the matrices with both payoffs shown, we see that the highlighted points are equilibrium points.

No Testing (Probabilities)

		Carter	
		d	**n**
Rogers	**d**	(0.5, 0.5)	(1, 0)
	n	(0, 1)	(0.5, 0.5)

IOC Testing (Expected Payoff)

		Carter	
		d	**n**
Rogers	**d**	$\left(-\frac{b}{2}, -\frac{b}{2}\right)$	$\left(-\frac{b}{2}, 0\right)$
	n	$\left(0, -\frac{b}{2}\right)$	$(0, 0)$

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No Testing (Probabilities)

		Carter	
		d	**n**
Rogers	**d**	(0.5, 0.5)	(1, 0)
	n	(0, 1)	(0.5, 0.5)

IOC Testing (Expected Payoff)

		Carter	
		d	**n**
Rogers	**d**	$\left(-\frac{b}{2}, -\frac{b}{2}\right)$	$\left(-\frac{b}{2}, 0\right)$
	n	$\left(0, -\frac{b}{2}\right)$	$(0, 0)$

- In particular, we see that with testing the incentive for both swimmers switches from taking the drugs to not taking the drugs.

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How de we find Equilibrium Points

An Equilibrium Point is Stable and once it is reached, it will generally persist through repeated playing of the game. When a game is not at an equilibrium point, at least one player has an incentive to change strategies, such a point is called unstable. A game may have a unique equilibrium point, more than one equilibrium point or no equilibrium points.

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- If the reduced payoff matrix has just a single strategy for both players, then this combination of strategies is an equilibrium point in the original and the reduced payoff matrix.
  For our previous example the red rows and columns shown below are dominated and are removed in the reduced matrices:

No Testing (Probabilities)

		Carter	
		d	**n**
Rogers	**d**	(0.5, 0.5)	(1, 0)
	n	(0, 1)	(0.5, 0.5)

IOC Testing (Expected Payoff)

		Carter	
		d	**n**
Rogers	**d**	$\left(-\frac{b}{2}, -\frac{b}{2}\right)$	$\left(-\frac{b}{2}, 0\right)$
	n	$\left(0, -\frac{b}{2}\right)$	$(0, 0)$

A systematic method to find Equilibrium Points

Not all reduced payoff matrices will have just a single strategy for both players. We can find the equilibrium points of a payoff matrix systematically as follows:

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- and the maximum payoff for the column player in each row

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- We compute the maximum payoff for the row player in each column
- and the maximum payoff for the column player in each row
- if a combination of strategies (a matrix entry) simultaneously give the maximum for the row player in its column and the maximum for the column player in its row, then this is an equilibrium point.

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- Example: Rose and Colin are playing a game where each has three strategies with payoff matrix is shown below:

	C1	**C2**	**C3**
R1	(6. 4)	(7, 1)	(8, 6)
R2	(1, 2)	(9, 5)	(4, 7)
R3	(8, 8)	(6, 2)	(3, 3)

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Max. R	8	9	8

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	C1	**C2**	**C3**	Max. C
R1	(6. 4)	(7, 1)	(8, 6)	6
R2	(1, 2)	(9, 5)	(4, 7)	7
R3	(8, 8)	(6, 2)	(3, 3)	8
Max. R	8	9	8	

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- Example: Rose and Colin are playing a game where each has three strategies with payoff matrix is shown below:

	C1	**C2**	**C3**	Max. C
R1	(6. 4)	(7, 1)	(8, 6)	6
R2	(1, 2)	(9, 5)	(4, 7)	7
R3	(8, 8)	(6, 2)	(3, 3)	8
Max. R	8	9	8	

- We see that there are equilibrium points at $R1C3$ and at $R3C1$.

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Dominated Strategies and reduced matrices in a Constant Sum Game

For a constant sum game or a zero-sum game, we just write the payoff for the row player since we can deduce the payoff for the column player from it.

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For a constant sum game or a zero-sum game, we just write the payoff for the row player since we can deduce the payoff for the column player from it.

- In this case a dominated strategy for the row player corresponds to a row where the entries are less than or equal to the corresponding entries in another row (the dominating strategy).

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Dominated Strategies and reduced matrices in a Constant Sum Game

For a constant sum game or a zero-sum game, we just write the payoff for the row player since we can deduce the payoff for the column player from it.

- In this case a dominated strategy for the row player corresponds to a row where the entries are less than or equal to the corresponding entries in another row (the dominating strategy).
- A dominated strategy for the column player corresponds to a column whose entries are greater than or equal (since the entries are payoffs for the row player) to the corresponding entries in another column (the dominating column).

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Dominated Strategies and reduced matrices in a Constant Sum Game

A (hypothetical) baseball pitcher throws three pitches, a fastball, a slider and a change-up. We might use the expected number of runs the batter creates in each situation as the payoff here. For any given pitch, the batter's performance is better if he anticipates the pitch (correctly). Lets assume that the batter has four possible strategies, To anticipate either a fastball, a slider or a change-up or not to anticipate any pitch.

		Pitcher		
		Fastball	**Change-up**	Slider
Batter	**Fastball**	0.3	0.3	0.35
	Change-up	0.25	0.4	0.4
	Slider	0.2	0.39	0.45
	None	0.3	0.39	0.4

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		Pitcher		
		Fastball	**Change-up**	Slider
Batter	**Fastball**	0.3	0.3	0.35
	Change-up	0.25	0.4	0.4
	Slider	0.2	0.39	0.45
	None	0.3	0.39	0.4

- The strategy "Slider" is dominated for the pitcher since the payoff for R is greater (or equal) no matter which strategy R uses when the pitcher pitches a Slider. We can thus reduce the payoff matrix by eliminating this strategy for C.

.

Dominated Strategies and reduced matrices in a Constant Sum Game

The reduced matrix looks like this:

		Pitcher	
		Fastball	**Change-up**
Batter	**Fastball**	0.3	0.3
	Change-up	0.25	0.4
	Slider	0.2	0.39
	None	0.3	0.39

.

Dominated Strategies and reduced matrices in a Constant Sum Game

The reduced matrix looks like this:

		Pitcher	
		Fastball	**Change-up**
Batter	**Fastball**	0.3	0.3
	Change-up	0.25	0.4
	Slider	0.2	0.39
	None	0.3	0.39

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Dominated Strategies and reduced matrices in a Constant Sum Game

The reduced matrix looks like this:

		Pitcher	
		Fastball	**Change-up**
Batter	**Fastball**	0.3	0.3
	Change-up	0.25	0.4
	Slider	0.2	0.39
	None	0.3	0.39

- We can further reduce this matrix by striking out the dominated strategies for R.

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Dominated Strategies and reduced matrices in a Constant Sum Game

The reduced matrix looks like this:

		Pitcher	
		Fastball	**Change-up**
Batter	**Fastball**	0.3	0.3
	Change-up	0.25	0.4
	Slider	0.2	0.39
	None	0.3	0.39

- We can further reduce this matrix by striking out the dominated strategies for the batter.
- The strategy "Anticipate Slider" for the batter has a lower payoff than that of "Anticipate Change-Up" no matter which pitch is thrown.

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Dominated Strategies and reduced matrices in a Constant Sum Game

The reduced matrix looks like this:

		Pitcher	
		Fastball	**Change-up**
Batter	**Fastball**	0.3	0.3
	Change-up	0.25	0.4
	Slider	0.2	0.39
	None	0.3	0.39

- We can further reduce this matrix by striking out the dominated strategies for the batter.
- The strategy "Anticipate Slider" for the batter has a lower payoff than that of "Anticipate Change-Up" no matter which pitch is thrown.
- Thus we can discard the strategy "Anticipate Slider" for the batter since it would be foolish for the batter to use this strategy and we are assuming that neither player is foolish.

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Dominated Strategies and reduced matrices in a Constant Sum Game

The (further) reduced matrix looks like this:

		Pitcher	
		Fastball	**Change-up**
Batter	**Fastball**	0.3	0.3
	Change-up	0.25	0.4
	None	0.3	0.39

.

Dominated Strategies and reduced matrices in a Constant Sum Game

		Pitcher	
		Fastball	**Change-up**
Batter	**Fastball**	0.3	0.3
	Change-up	0.25	0.4
	None	0.3	0.39

- Now in this matrix, we see that the strategy "Change-Up" is a dominated strategy for the pitcher, because no matter what the batter anticipates, the payoff is greater (or equal) for the pitcher if the pitch a fastball instead of a Change-Up.
- The new reduced matrix looks like :

		Pitcher
		Fastball
Batter	**Fastball**	0.3
	Change-up	0.25
	None	0.3

.

Dominated Strategies and reduced matrices in a Constant Sum Game

Striking out dominated strategies for the batter, we get:

		Pitcher
		Fastball
Batter	**Fastball**	0.3
	Change-up	0.25
	None	0.3

$\longrightarrow$

		Pitcher
		Fastball
Batter	**Fastball**	0.3
	None	0.3

.

Dominated Strategies and reduced matrices in a Constant Sum Game

Striking out dominated strategies for the batter, we get:

		Pitcher
		Fastball
Batter	**Fastball**	0.3
	Change-up	0.25
	None	0.3

$\longrightarrow$

		Pitcher
		Fastball
Batter	**Fastball**	0.3
	None	0.3

- This is the reduced matrix since there are no dominated strategies.

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Equilibrium Points in a Constant Sum Game

In the case of zero-sum or constant-sum games an equilibrium point is called a saddle point. The value of the pay-off matrix at that position is called the value of the game.

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In the case of zero-sum or constant-sum games an equilibrium point is called a saddle point. The value of the pay-off matrix at that position is called the value of the game.

- If an equilibrium point exists in the game, it occurs at a point which is simultaneously the minimum in its row and the maximum in its column (since neither player has an incentive to change strategy at that point).

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Equilibrium Points in a Constant Sum Game

In the case of zero-sum or constant-sum games an equilibrium point is called a saddle point. The value of the pay-off matrix at that position is called the value of the game.

- If an equilibrium point exists in the game, it occurs at a point which is simultaneously the minimum in its row and the maximum in its column (since neither player has an incentive to change strategy at that point).
- Although the equilibrium point may not be unique, if there are multiple equilibrium points for the game, all will give the same value (payoff for the row player).

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Equilibrium Points in a Constant Sum Game

In the case of zero-sum or constant-sum games an equilibrium point is called a saddle point. The value of the pay-off matrix at that position is called the value of the game.

- If an equilibrium point exists in the game, it occurs at a point which is simultaneously the minimum in its row and the maximum in its column (since neither player has an incentive to change strategy at that point).
- Although the equilibrium point may not be unique, if there are multiple equilibrium points for the game, all will give the same value (payoff for the row player).
- To find equilibrium points for constant-sum games, we can calculate the minimum for each row and the maximum for each column and see if any entry in the matrix simultaneously gives the minimum in its row and the maximum in its column.

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Saddle Points in a Constant Sum Game, Example

For our previous example we apply our method to find the saddle points. We calculate the minimum in each row and the maximum in each column.

		Pitcher			
		Fastball	**Change-up**	**Slider**	**Min.**
Batter	**Fastball**	0.3	0.3	0.35	0.3
	Change-up	0.25	0.4	0.4	0.25
	Slider	0.2	0.39	0.45	0.2
	None	0.3	0.39	0.4	0.3
	Max.	0.3	0.4	0.45	

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Saddle Points in a Constant Sum Game, Example

For our previous example we apply our method to find the saddle points. We calculate the minimum in each row and the maximum in each column.

		Pitcher			
		Fastball	**Change-up**	**Slider**	**Min.**
Batter	**Fastball**	0.3	0.3	0.35	0.3
	Change-up	0.25	0.4	0.4	0.25
	Slider	0.2	0.39	0.45	0.2
	None	0.3	0.39	0.4	0.3
	Max.	0.3	0.4	0.45	

.

Saddle Points in a Constant Sum Game, Example

For our previous example we apply our method to find the saddle points. We calculate the minimum in each row and the maximum in each column.

		Pitcher			
		Fastball	**Change-up**	**Slider**	**Min.**
Batter	**Fastball**	0.3	0.3	0.35	0.3
	Change-up	0.25	0.4	0.4	0.25
	Slider	0.2	0.39	0.45	0.2
	None	0.3	0.39	0.4	0.3
	Max.	0.3	0.4	0.45	

- We see that there are <u>two saddle points,</u> One where the pitcher pitches a fastball and the batter anticipates a fastball, the other where the pitcher pitches a fastball and the batter does not anticipate any serve.
- The <u>value</u> of the game is 0.3. This is the payoff for the batter at all of the saddle points.

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Saddle Points in a Constant Sum Game, Example

In the example from Winston in the last section, the football team on offense had the options of running or passing the ball and the team on defense could choose a run defense or a pass defense. We got the following pay-off matrix for the team on offense using expected gain in yards for each situation:

		Defense	
		Run Defense	**Pass Defense**
Offense	**Run**	-5	5
	Pass	10	0

(a) Does this matrix have a saddle point? .

Saddle Points in a Constant Sum Game, Example

In the example from Winston in the last section, the football team on offense had the options of running or passing the ball and the team on defense could choose a run defense or a pass defense. We got the following pay-off matrix for the team on offense using expected gain in yards for each situation:

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- We see that there are no saddle points, since there is no entry in the matrix which is the minimum in its row and the maximum in its column.

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Choosing a Strategy for a Zero-Sum Game

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- Their opponent will play intelligently and wishes to maximize their own payoff.
- **Note** that in a zero sum game The column player maximizes their payoff by minimizing the row players payoff.
- For a zero-sum game or a constant-sum game with a saddle point, if the above assumptions hold and the game is played repeatedly the play will eventually stabilize at the saddle point.

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- The value of the game is the long run expected payoff for the row player R when the game is played repeatedly since neither player will have any incentive to choose a different strategy than the one at the saddle point.

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- The value of the game is the long run expected payoff for the row player R when the game is played repeatedly since neither player will have any incentive to choose a different strategy than the one at the saddle point.
- Above, we saw that a saddle point does not always exist in a zero-sum game. In this case it turns out that a mixed strategy is better to maximize long run expected payoff.

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		Fastball	**Change-up**	**Slider**	**Min.**
Batter	**Fastball**	0.3	0.3	0.35	0.3
	Change-up	0.25	0.4	0.4	0.25
	Slider	0.2	0.39	0.45	0.2
	None	0.3	0.39	0.4	0.3
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- Since this is a zero sum game the expected payoff for the pitcher is -0.3.

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On the other hand, our example from Football did not have a saddle point and therefore a fixed strategy was not the best option for either player.

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- The row player's mixed strategy is represented as a row $\begin{pmatrix} p_1, p_2 \end{pmatrix}$ and the column player's mixed strategy is represented as a column $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$

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Mixed Strategy

Recall our example from fencing where both players had the option of attacking(A) or holding back (H) at the beginning of each bout with the following pay-off matrix showing the expected number of points for Rhonda for each situation:

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		A	**H**
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- If Rhonda plays the mixed strategy of $\left(\ 0.5, 0.5\ \right)$, it means that she attacks off the line 50 percent of the time and she holds back 50% of the time. she does this in an unpredictable way, so that her opponent does not know whether she will attack or hold back at the beginning of the next bout.

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- If Cathy plays $\begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$, this means that Cathy attacks off the line 70% of the time and she hold back 30% of the time and her choice for the next bout cannot be predicted by her opponent.

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- **Note** that a fixed strategy can be represented as a special case of a mixed strategy. If Rhonda plays $(1, 0)$ above, it means that she always plays strategy A.

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- and for the column player, C, the strategy which minimizes R's expected payoff (maximizes C's expected payoff) is the preferred one.
- When both players have only two strategies, calculating the expected payoff for R when you know the strategies for both players is relatively easy. When more strategies are involved it is best to calculate with matrix multiplication.

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- Notice that the payoff for R is a random variable, X, and its value depends on which of the four situations occurs.

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Choice	X = Pay-off for R	Probability	$XP(X)$
$R1C1$	a	$(p_1)(q_1)$	$a(p_1)(q_1)$
$R1C2$	b	$(p_1)(q_2)$	$b(p_1)(q_2)$
$R2C1$	c	$(p_2)(q_1)$	$c(p_2)(q_1)$
$R2C2$	d	$(p_2)(q_2)$	$d(p_2)(q_2)$
			$E(X) = (ap_1 + cp_2)(q_1) +$ $+(bp_1 + dp_2)(q_2)$

Example: Expected Payoff for mixed strategies

Consider the example of a zero-sum game from fencing above with payoff matrix :

		Cathy	
		A	H
Rhonda	A	0.5	-0.2
	H	-0.3	0.5

Assume that Rhonda(R) plays $(0.5, 0.5)$ and Cathy(C) plays $\begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$, calculate the expected payoff for R. What is the expected payoff for C?

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Choice	Pay-off for R	Probability	$XP(X)$
$RACA$	0.5	$(0.5)(0.7) = 0.35$	0.175
$RACH$	-0.2	$(0.5)(0.3) = 0.15$	-0.03
$RHCA$	-0.3	$(0.5)(0.7) = 0.35$	-0.105
$RHCH$	0.5	$(0.5)(0.3) = 0.15$	0.075
			$E(X) = 0.115$

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			$E(X) = 0.115$

- The expected payoff for R is $0.115 \approx$ *ave.*# points per game for R if both players fence many times with the above strategies.
- The expected payoff for C is $-0.115 \approx$ *ave.*# points per game for C if both players fence many times with the above strategies.

Example: Expected Payoff for different mixed strategies

Suppose now that C continues to play the strategy $\begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$, bur R switches to the strategy $(0.2, 0.8)$, what is the expected payoff for R?

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$RACA$	0.5	$(0.2)(0.7) = 0.14$	0.07
$RACH$	-0.2	$(0.2)(0.3) = 0.06$	-0.012
$RHCA$	-0.3	$(0.8)(0.7) = 0.56$	-0.168
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- On the other hand if R plays $(0.5, 0.5)$, R's expected payoff will be 0.115,

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- On the other hand if R plays $(0.5, 0.5)$, R's expected payoff will be 0.115,
- thus if R is choosing between strategies $(0.2, 0.8)$ and $(0.5, 0.5)$, the latter is the better strategy.

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- We will then use this information to pick the strategy for R. Given that C will hold R's expected payoff to a minimum, we will choose the strategy for R which maximizes these minima. This is often called minimax theory for obvious reasons.

Best Counterstrategy for C

From our previous calculations, if R plays any strategy $(p, 1-p)$ and C plays $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ with payoff matrix:

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Choice	X = Pay-off for R	Probability	$XP(X)$
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- For any given value of p, let $J = (ap + c(1-p))$ and let $K = (bp + d(1-p))$.
- Then the expected payoff for R is $\boxed{E(X) = Jq_1 + Kq_2.}$

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Our bottom line is that the expected payoff for R is $\boxed{E(X) = Jq_1 + Kq_2}$ for two numbers J and K, when R plays any strategy $(p, 1-p)$ for some fixed value of p and C plays $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$

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- Thus since Case 1 or Case 2 must happen, the best counterstrategy for C is always a fixed strategy.

Strategy Lines and R's optimal Mixed Strategy

Lets consider an example where the payoff matrix for the row player R is given by:

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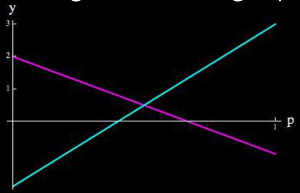

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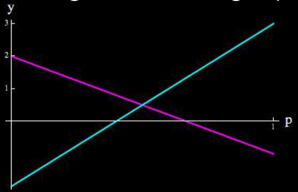

- Their equations are $y = K = 3p - 2(1-p) = 5p - 2$ and $y = J = -p + 2(1-p) = 2 - 3p$.

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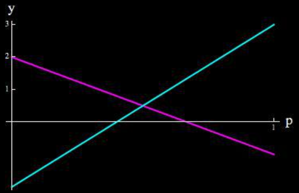

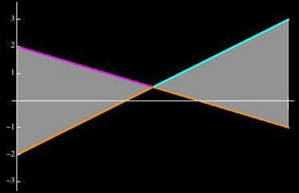

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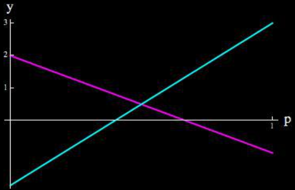

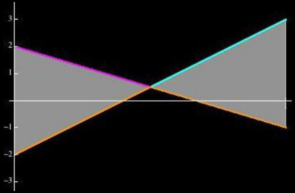

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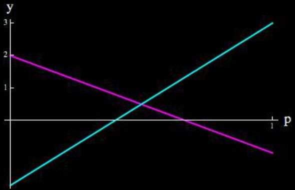

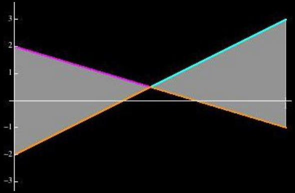

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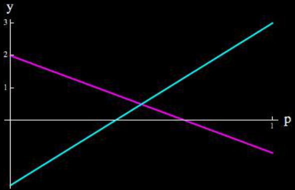

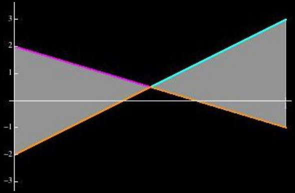

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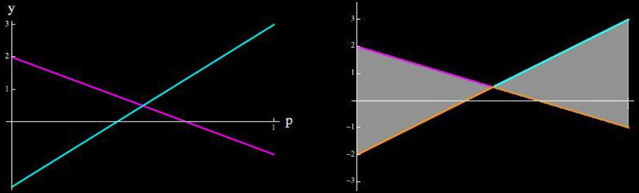

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- This minimum possible expected payoff for R for any given value of p will be on the lowest of the two lines above p and the maximum will be on the highest of the two lines shown.
- Our assumptions are that C responds appropriately and quickly to reduce R's payoff to a minimum and whichever value of p that R chooses, R's expected payoff will be on the lower of the two lines (on the lines highlighted in orange).

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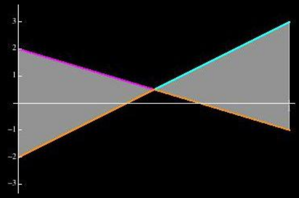

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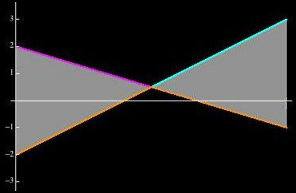

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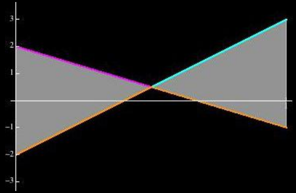

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- To find the value of p where the line $y = 2 - 3p$ meets the line $y = 5p - 2$, we set the y values equal to each other to get $2 - 3p = 5p - 2$. This gives us that $4 = 8p$ or $p = 1/2$.

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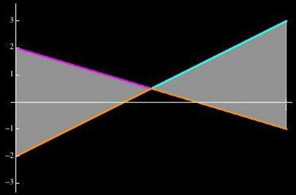

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- Thus the best strategy for R is $(1/2, 1/2)$ in this case.

Optimal Mixed Strategy, General Case

If R's payoff matrix has a saddle point, then the lines might not meet or may meet when $p = 0$ or when $p = 1$. Otherwise (in the case where the strategy matrix is reduced), we can solve p at the point where both lines meet.

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the optimal mixed strategy for R is given by $(p, 1-p)$ where p is given by the solution to the equation $ap + c(1-p) = bp + d(1-p)$ that is when $\boxed{p = \frac{d-c}{(a+d)-(b+c)}}$.

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- Using these optimal mixed strategies for both players, we get that the value of the game is given by

$$\boxed{\nu = \frac{ad-bc}{(a+d)-(b+c)}.}$$

Optimal Mixed Strategy, Football Example

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- The value of the game is 2.5, meaning that if both players play optimal strategies, the long run average yards gained per play by the offense will be 2.5 yards.

Optimal Mixed Strategy, Interpretation

		Defense	
		Run Defense	Pass Defense
Offense	Run	-5	5
	Pass	10	0

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- The value of the game is 2.5, meaning that if both players play optimal strategies, the long run average yards gained per play by the offense will be 2.5 yards.
- The optimal strategies give an equilibrium point, in that neither player has an incentive to change strategy if the opponent continues with their optimal strategy.

Theorem of John Von Neumann

The existence of such an equilibrium holds true for all two person zero-sum games:

Minimax Theorem: John Von Neumann For every zero sum game, there is a number ν for value and particular mixed strategies for both players such that

1. The expected payoff to the row player will be at least ν if the row player plays his or her particular mixed strategy, no matter what mixed strategy the column player plays.
2. The expected payoff to the row player will be at most ν if the column player plays his or her particular mixed strategy, no matter what strategy the row player chooses.

the number ν is called the value of the game and represents the expected advantage to the row player (a disadvantage if ν is negative).

If both players play the strategies from the theorem, the system will be in equilibrium, since neither player should be able to increase their payoff by unilaterally changing their strategy. Thus the long run expected payoff for R will be ν and this is the **value** of the game.

Optimal Mixed Strategy, Fencing Example

In our example from fencing where both players had the option of attacking(A) or holding back (H) at the beginning of each bout we also had a reduced payoff matrix for the row player:

		Cathy(C)	
		A	**H**
Rhonda(R)	**A**	0.5	-0.2
	H	-0.3	0.5

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- The value of the game is $\nu = \frac{ad-bc}{(a+d)-(b+c)}. = \frac{(0.5)(0.5)-(-0.3)(-0.3)}{1.5} = \frac{0.16}{1.5} = \frac{1.6}{15} \approx 0.11$

Optimal Mixed Strategy, Constant Sum Game

For a constant sum game the calculations are the same for the optimal mixed strategy for both players and the value of the game, ν (which is the expected payoff for R).

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- If the payoffs for R and C add to K, then the long run expected payoff for C is $K - \nu$ (as opposed to $-\nu$ for zero sum games).

Example, Constant Sum Game, Basketball

Recall our example of possible endgame strategies for basketball due to Ruminski, where the payoff for the offense was given as the probability of a win for the offensive team. This is a **Constant sum game** since the probabilities add to 1.

		Defending	Team
		Defend 2	**Defend 3**
Offense	**Shoot 2**	0.178	0.312
	Shoot 3	0.5	0.228

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- This is a reduced payoff matrix, so we can apply our formulas. The optimal mixed strategy for R is given by $(p, 1-p)$ where $p = \frac{d-c}{(a+d)-(b+c)}. = \frac{0.228-0.5}{(0.178+0.228)-(0.312+0.5)} = \frac{-0.272}{-0.406} \approx 0.67$. R's optimal mixed strategy is (approximately) $(0.67, 0.33)$.

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- The optimal mixed strategy for C is given by $\begin{pmatrix} q \\ 1-q \end{pmatrix}$ where $q = \frac{d-b}{(a+d)-(b+c)} = \frac{0.228-0.312}{-0.406} = \frac{-0.084}{-0.406} \approx 0.21$. C's optimal mixed strategy is given by $\begin{pmatrix} 0.21 \\ 0.79 \end{pmatrix}$

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- The value of the game is $\nu = \frac{ad-bc}{(a+d)-(b+c)}. = \frac{(0.178)(0.228)-(0.5)(0.312)}{-0.406} = \frac{-0.115}{-0.406} \approx 0.28$.

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- The value of the game 0.28. This means that if both players play optimally and the scenario is repeated many times, the Offense can expect to win about 28% of the time.

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- This means that if both players play optimally and the scenario is repeated many times, the Offense can expect to win about 28% of the time (and this is the best that they can do if their opponents play optimally).
- This expected payoff for the defense is $1 - \nu = 1 - 0.28 = 0.72$. This means that if both play optimally, then the defense can expected to win about 72% of the time in when this scenario arises.

Reducing a matrix: Baseball

Lets look at a batter vs. Pitcher scenario again where the payoff matrix is given below (different from the previous batter/pitcher matrix):

		Pitcher		
		Fastball	**Change-up**	**Slider**
Batter	**Fastball**	0.38	0.37	0.39
	Change-up	0.25	0.4	0.41
	Slider	0.35	0.32	0.45
	None	0.38	0.3	0.42

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- We cannot use our formulas directly here, because this is a three by three matrix.
- We can however reduce the matrix to a reduced two by two matrix, by striking out dominated strategies. The strategy "slider" is dominated for the column player and when we strike that out, the strategies "Anticipate Slider" and "Anticipate None" are dominated for the batter in the reduced matrix.

Reducing a matrix: Baseball

The reduced matrix is

		Pitcher	
		Fastball	**Change-up**
Batter	**Fastball**	0.38	0.37
	Change-up	0.25	0.4

Reducing a matrix: Baseball

The reduced matrix is

		Pitcher	
		Fastball	**Change-up**
Batter	**Fastball**	0.38	0.37
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- We can now use our formulas to find the optimal strategy for both players. The optimal mixed strategy for R is given by $(p, 1-p)$ where $p = \frac{d-c}{(a+d)-(b+c)}. = \frac{0.4-0.25}{(0.38+0.4)-(0.37+0.25)} = \frac{0.15}{0.16} = \frac{15}{16}$. R's optimal mixed strategy is (approximately) $(15/16, 1/16)$.

Reducing a matrix: Baseball

The reduced matrix is

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		Fastball	Change-up
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- The value of the game is $\nu = \frac{ad-bc}{(a+d)-(b+c)}. = \frac{(0.38)(0.4)-(0.25)(0.37)}{0.16} = \frac{0.0595}{0.16} \approx 0.37$.

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- Since the payoffs are given as expected runs created by the batter, the average number of runs created per pitch for the batter in the long run will be 0.37 if both players play optimally.